A SEPARABLE DEFORMATION OF THE QUATERNION GROUP ALGEBRA

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Abstract. The Donald-Flanigan conjecture asserts that for any finite group G and any field k, the group algebra kG can be deformed to a separable algebra. The minimal unsolved instance, namely the quaternion group Q_8 over a field k of characteristic 2 was considered as a counterexample. We present here a separable deformation of kQ_8. In a sense, the conjecture for any finite group is open again.

1. Introduction

In their paper [1], J.D. Donald and F.J. Flanigan conjectured that any group algebra kG of a finite group G over a field k can be deformed to a semisimple algebra even in the modular case, namely where the order of G is not invertible in k. A more customary formulation of the Donald-Flanigan (DF) conjecture is by demanding that the deformed algebra $[kG]_t$ should be separable, i.e. it remains semisimple when tensored with the algebraic closure of its base field. If, additionally, the dimensions of the simple components of $[kG]_t$ are in one-to-one correspondence with those of the complex group algebra $\mathbb{C}G$, then $[kG]_t$ is called a *strong* solution to the problem.

The DF conjecture was solved for groups G which have either a cyclic p-Sylow subgroup over an algebraically closed field [11] or a normal abelian p-Sylow subgroup [5] where $p=\mathrm{char}(k)$, and for all but six reflection groups in any characteristic [6, 7, 10]. In [4], it is claimed that the group algebra kQ_8, where

$$Q_8 = \langle\, \sigma, \tau | \sigma^4 = 1, \tau\sigma = \sigma^3\tau, \sigma^2 = \tau^2 \,\rangle$$

is the quaternion group of order 8 and k a field of characteristic 2, does not admit a separable deformation. This result allegedly gives a counterexample to the DF conjecture. However, as observed by M. Schaps, the proof apparently contains an error (see §7).

The aim of this note is to present a separable deformation of kQ_8, where k is any field of characteristic 2, reopening the DF conjecture.

2. Preliminaries

Let $k[[t]]$ be the ring of formal power series over k, and let $k((t))$ be its field of fractions. Recall that the deformed algebra $[kG]_t$ has the same underlying $k((t))$-vector space as $k((t)) \otimes_k kG$, with multiplication defined on basis elements

$$g_1 * g_2 := g_1 g_2 + \sum_{i\geq 1} \Psi_i(g_1, g_2)t^i, \quad g_1, g_2 \in G \tag{2.1}$$

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and extended $k((t))$-linearly (such that t is central). Here g_1g_2 is the group multiplication. The functions $\Psi_i : G\times G \to kG$ satisfy certain cohomological conditions induced by the associativity of $[kG]_t$ [3, §1 ; §2].

Note that the set of equations (2.1) determines a multiplication on the free $k[[t]]$-module Λ_t spanned by the elements $\{g\}_{g\in G}$ such that $kG \simeq \Lambda_t/\langle t\Lambda_t\rangle$ and $[kG]_t \simeq \Lambda_t \otimes_{k[[t]]} k((t))$. In a more general context, namely over a domain R which is not necessarily local, the R-module Λ_t which determines the deformation, is required only to be *flat* rather than free [2, §1].

In what follows, we shall define the deformed algebra $[kG]_t$ by using generators and relations. These will implicitly determine the set of equations (2.1).

3. Sketch of the construction

Consider the extension

$$[\beta] : 1 \to C_4 \to Q_8 \to C_2 \to 1, \tag{3.1}$$

where $C_2 = \langle\, \bar\tau\, \rangle$ acts on $C_4 = \langle\, \sigma\, \rangle$ by

$$\begin{array}{rccc} \eta: & C_2 & \to & \mathrm{Aut}(C_4) \\ \eta(\bar\tau): & \sigma & \mapsto & \sigma^3(=\sigma^{-1}), \end{array}$$

and the associated 2-cocycle $\beta : C_2\times C_2 \to C_4$ is given by

$$\beta(1,1) = \beta(1,\bar\tau) = \beta(\bar\tau,1) = 1, \beta(\bar\tau,\bar\tau) = \sigma^2.$$

The group algebra kQ_8 (k any field) is isomorphic to the quotient $kC_4[y;\eta]/\langle\, q(y)\,\rangle$, where $kC_4[y;\eta]$ is a skew polynomial ring [9, §1.2], whose indeterminate y acts on the ring of coefficients kC_4 via the automorphism $\eta(\bar\tau)$ (extended linearly) and where

$$q(y) := y^2 - \sigma^2 \in kC_4[y;\eta] \tag{3.2}$$

is central. The above isomorphism is established by identifying τ with the indeterminate y.

Suppose now that $\mathrm{Char}(k) = 2$. The deformed algebra $[kQ_8]_t$ is constructed as follows.

In §4.1 the subgroup algebra kC_4 is deformed to a separable algebra $[kC_4]_t$ which is isomorphic to $K \oplus k((t)) \oplus k((t))$, where K is a separable field extension of $k((t))$ of degree 2.

The next step (§4.2) is to construct an automorphism η_t of $[kC_4]_t$ which agrees with the action of C_2 on kC_4 when specializing $t = 0$. This action fixes all three primitive idempotents of $[kC_4]_t$. By that we obtain the skew polynomial ring $[kC_4]_t[y;\eta_t]$.

In §5 we deform $q(y) = y^2 + \sigma^2$ to $q_t(y)$, a separable polynomial of degree 2 in the center of $[kC_4]_t[y;\eta_t]$.

By factoring out the two-sided ideal generated by $q_t(y)$, we establish the deformation

$$[kQ_8]_t := [kC_4]_t[y;\eta_t]/\langle\, q_t(y)\,\rangle.$$

In §6 we show that $[kQ_8]_t$ as above is separable. Moreover, passing to the algebraic closure $\overline{k((t))}$ we have

$$[kQ_8]_t \otimes_{k((t))} \overline{k((t))} \simeq \bigoplus_{i=1}^{4} \overline{k((t))} \oplus M_2(\overline{k((t))}).$$

This is a strong solution to the DF conjecture since its decomposition to simple components is the same as

$$\mathbb{C}Q_8 \simeq \bigoplus_{i=1}^{4} \mathbb{C} \oplus M_2(\mathbb{C}).$$

4. A Deformation of $kC_4[y;\eta]$

4.1. We begin by constructing $[kC_4]_t$, $C_4 = \langle\, \sigma \,\rangle$. Recall that

$$kC_4 \simeq k[x]/\langle\, x^4+1 \,\rangle$$

by identifying σ with $x+\langle\, x^4+1 \,\rangle$. We deform the polynomial x^4+1 to a separable polynomial $p_t(x)$ as follows.

Let $k[[t]]^*$ be the group of invertible elements of $k[[t]]$ and denote by

$$U := \{1+zt | z \in k[[t]]^*\}$$

its subgroup of 1-units (when $k = \mathbb{F}_2$, U is equal to $k[[t]]^*$).

Let

$$a \in k[[t]] \setminus k[[t]]^*$$

be a non-zero element, and let

$$b, c, d \in U, (c \neq d),$$

such that

$$\pi(x) := x^2 + ax + b$$

is an irreducible (separable) polynomial in $k((t))[x]$. Let

$$p_t(x) := \pi(x)(x+c)(x+d) \in k((t))[x].$$

Then the quotient $k((t))[x]/\langle\, p_t(x) \,\rangle$ is isomorphic to the direct sum $K \oplus k((t)) \oplus k((t))$, where $K := k((t))[x]/\langle\, \pi(x) \,\rangle$. The field extension $K/k((t))$ is separable and of dimension 2.

Note that $p_{t=0}(x) = x^4+1$ and that only lower order terms of the polynomial were deformed. Hence, the quotient $k[[t]][x]/\langle\, p_t(x) \,\rangle$ is $k[[t]]$-free and $k((t))[x]/\langle\, p_t(x) \,\rangle$ indeed defines a deformation $[kC_4]_t$ of $kC_4 \simeq k[x]/\langle\, x^4+1 \,\rangle$. The new multiplication $\sigma^i * \sigma^j$ of basis elements (2.1) is determined by identifying σ^i with $\bar{x}^i := x^i + \langle\, p_t(x) \,\rangle$. We shall continue to use the term $\bar{x}$ in $[kC_4]_t$ rather than σ.

Assume further that there exists $w \in k[[t]]$ such that

$$(x+w)(x+c)(x+d) = x\pi(x) + a \tag{4.1}$$

(see example 4.3). Then $K \simeq ([kC_4]_t)e_1$, where

$$e_1 = \frac{(\bar{x}+w)(\bar{x}+c)(\bar{x}+d)}{a}. \tag{4.2}$$

The two other primitive idempotents of $[kC_4]_t$ are

$$e_2 = \frac{c(\bar{x}+d)\pi(\bar{x})}{a(c+d)}, \quad e_3 = \frac{d(\bar{x}+c)\pi(\bar{x})}{a(c+d)}. \tag{4.3}$$

4.2. Let

$$\eta_t : k((t))[x] \to k((t))[x]$$

be an algebra endomorphism determined by its value on the generator x as follows.

$$\eta_t(x) := x\pi(x) + x + a. \tag{4.4}$$

We compute $\eta_t(\pi(x))$, $\eta_t(x+c)$ and $\eta_t(x+d)$:

$$\begin{aligned}\eta_t(\pi(x)) = \eta_t(x)^2 + a\eta_t(x) + b &= x^2\pi(x)^2 + x^2 + a^2 + ax\pi(x) + ax + a^2 + b \\ &= \pi(x)(x^2\pi(x) + ax + 1).\end{aligned}$$

By (4.1),

$$\eta_t(\pi(x)) = \pi(x) + x(x+w)p_t(x) \in \langle\, \pi(x)\,\rangle. \tag{4.5}$$

Next,

$$\eta_t(x+c) = x\pi(x) + x + a + c.$$

By (4.1),

$$\eta_t(x+c) = (x+c)[(x+w)(x+d)+1] \in \langle\, x+c\,\rangle. \tag{4.6}$$

Similarly,

$$\eta_t(x+d) = (x+d)[(x+w)(x+c)+1] \in \langle\, x+d\,\rangle. \tag{4.7}$$

By (4.5), (4.6) and (4.7), we obtain that $\eta_t(p_t(x)) \in \langle\, p_t(x)\,\rangle$, and hence η_t induces an endomorphism of $k((t))[x]/\langle\, p_t(x)\,\rangle$ which we continue to denote by η_t. As can easily be verified, the primitive idempotents given in (4.2) and (4.3) are fixed under η_t:

$$\eta_t(e_i) = e_i, \quad i = 1, 2, 3, \tag{4.8}$$

whereas

$$\eta_t(\bar{x}e_1) = \eta_t(\bar{x})e_1 = (\bar{x}\pi(\bar{x}) + \bar{x} + a)e_1 = (\bar{x} + a)e_1. \tag{4.9}$$

Hence, η_t induces an automorphism of K of order 2 while fixing the two copies of $k((t))$ pointwise. Furthermore, one can easily verify that

$$\eta_{t=0}(\bar{x}) = \bar{x}^3.$$

Consequently, the automorphism η_t of $[kC_4]_t$ agrees with the automorphism $\eta(\bar{\tau})$ of kC_4 when $t = 0$. The skew polynomial ring

$$[kC_4]_t[y;\eta_t] = (k((t))[x]/\langle\, p_t(x)\,\rangle)[y;\eta_t]$$

is therefore a deformation of $kC_4[y;\eta]$.

Note that by (4.8), the idempotents $e_i, i = 1, 2, 3$ are central in $[kC_4]_t[y;\eta_t]$ and hence

$$[kC_4]_t[y;\eta_t] = \bigoplus_{i=1}^{3} [kC_4]_t[y;\eta_t]e_i. \tag{4.10}$$

4.3. **Example.** The following is an example for the above construction.
Put

$$a := \frac{t+t^2+t^3}{1+t}, b := 1+t^2+t^3, c := \frac{1}{1+t}, d := 1+t+t^2, w := t.$$

These elements satisfy equation (4.1):

$$\begin{aligned}(x+w)(x+c)(x+d) &= (x+t)(x+\frac{1}{1+t})(x+1+t+t^2) \\ = x^3 + \frac{t+t^2+t^3}{1+t}x^2 + (1+t^2+t^3)x + \frac{t+t^2+t^3}{1+t} &= x\pi(x)+a.\end{aligned}$$

The polynomial

$$\pi(x) = x^2 + \frac{t+t^2+t^3}{1+t}x + 1 + t^2 + t^3$$

does not admit roots in $k[[t]]/\langle\, t^2\,\rangle$, thus it is irreducible over $k((t))$.

5. A Deformation of $q(y)$

The construction of $[kQ_8]_t$ will be completed once the product $\bar{\tau} * \bar{\tau}$ is defined. For this purpose the polynomial $q(y)$ (3.2), which determined the ordinary multiplication τ^2, will now be developed in powers of t.

For any non-zero element $z \in k[[t]] \setminus k[[t]]^*$, let

$$q_t(y) := y^2 + z\bar{x}\pi(\bar{x})y + \bar{x}^2 + a\bar{x} \in [kC_4]_t[y;\eta_t]. \tag{5.1}$$

Decomposition of (5.1) with respect to the idempotents e_1, e_2, e_3 yields

$$q_t(y) = (y^2+b)e_1 + [y^2+zay+c(c+a)]e_2 + [y^2+zay+d(d+a)]e_3. \tag{5.2}$$

We now show that $q_t(y)$ is in the center of $[kC_4]_t[y;\eta_t]$:

First, the leading term y^2 is central since the automorphism η_t is of order 2. Next, by (4.8), the free term $be_1 + c(c+a)e_2 + d(d+a)e_3$ is invariant under the action of η_t and hence central. It is left to check that the term $za(e_2+e_3)y$ is central. Indeed, since e_2 and e_3 are η_t-invariant, then $za(e_2+e_3)y$ commutes both with $[kC_4]_t[y;\eta_t]e_2$ and $[kC_4]_t[y;\eta_t]e_3$. Furthermore, by orthogonality

$$za(e_2+e_3)y \cdot [kC_4]_t[y;\eta_t]e_1 = [kC_4]_t[y;\eta_t]e_1 \cdot za(e_2+e_3)y = 0,$$

and hence $za(e_2+e_3)y$ commutes with $[kC_4]_t[y;\eta_t]$.

Consequently, $\langle\, q_t(y)\,\rangle = q_t(y)[kC_4]_t[y;\eta_t]$ is a two-sided ideal.

Now, as can easily be deduced from (5.1),

$$q_{t=0}(y) = y^2 + \bar{x}^2 = q(y), \tag{5.3}$$

where the leading term y^2 remains unchanged. Then

$$[kQ_8]_t := [kC_4]_t[y;\eta_t]/\langle\, q_t(y)\,\rangle$$

is a deformation of kQ_8, identifying $\bar{\tau}$ with $\bar{y} := y + \langle\, q_t(y)\,\rangle$.

6. Separability of $[kQ_8]_t$

Finally, we need to prove that the deformed algebra $[kQ_8]_t$ is separable. Moreover, we prove that its decomposition to simple components over the algebraic closure of $k((t))$ resembles that of $\mathbb{C}Q_8$. By (4.10), we obtain

$$[kQ_8]_t = \bigoplus_{i=1}^{3} [kC_4]_t[y;\eta_t]e_i/\langle\, q_t(y)e_i \,\rangle. \tag{6.1}$$

We handle the three summands in (6.1) separately:

By (5.2),

$$[kC_4]_t[y;\eta_t]e_1/\langle\, q_t(y)e_1 \,\rangle \simeq K[y;\eta_t]/\langle\, y^2+b \,\rangle \simeq K^f * C_2.$$

The rightmost term is the *crossed product* of the group $C_2 := \langle\, \bar{\tau} \,\rangle$ acting faithfully on the field $K = [kC_4]_t e_1$ via η_t (4.9), with a twisting determined by the 2-cocycle $f : C_2 \times C_2 \to K^*$:

$$f(1,1) = f(1,\bar{\tau}) = f(\bar{\tau},1) = 1, \;\; f(\bar{\tau},\bar{\tau}) = b.$$

This is a central simple algebra over the subfield of invariants $k((t))$ [8, Theorem 4.4.1]. Evidently, this simple algebra is split by $\overline{k((t))}$, i.e.

$$[kC_4]_t[y;\eta_t]e_1/\langle\, q_t(y)e_1 \,\rangle \otimes_{k((t))} \overline{k((t))} \simeq M_2(\overline{k((t))}). \tag{6.2}$$

Next, since η_t is trivial on $[kC_4]_t e_2$, the skew polynomial ring $[kC_4]_t e_2[y;\eta_t]$ is actually an ordinary polynomial ring $k((t))[y]$. Again by (5.2),

$$[kC_4]_t[y;\eta_t]e_2/\langle\, q_t(y)e_2 \,\rangle \simeq k((t))[y]/\langle\, y^2+zay+c(c+a) \,\rangle.$$

Similarly,

$$[kC_4]_t[y;\eta_t]e_3/\langle\, q_t(y)e_3 \,\rangle \simeq k((t))[y]/\langle\, y^2+zay+d(d+a) \,\rangle.$$

The polynomials $y^2+zay+c(c+a)$ and $y^2+zay+d(d+a)$ are separable (since za is non-zero). Thus, both $[kC_4]_t[y;\eta_t]e_2/\langle\, q_t(y)e_2 \,\rangle$ and $[kC_4]_t[y;\eta_t]e_3/\langle\, q_t(y)e_3 \,\rangle$ are separable $k((t))$-algebras, and for $i=2,3$

$$[kC_4]_t[y;\eta_t]e_i/\langle\, q_t(y)e_i \,\rangle \otimes_{k((t))} \overline{k((t))} \simeq \overline{k((t))} \oplus \overline{k((t))}. \tag{6.3}$$

Equations (6.1), (6.2) and (6.3) yield

$$[kQ_8]_t \otimes_{k((t))} \overline{k((t))} \simeq \bigoplus_{i=1}^{4} \overline{k((t))} \oplus M_2(\overline{k((t))})$$

as required.

7. Acknowledgement

We wish to thank M. Schaps for pointing out to us that there is an error in the attempted proof in [4] that the quaternion group is a counterexample to the DF conjecture. Here is her explanation: The given relations for the group algebra are incorrect. Using the notation in pages 166-7 of [4], if $a = 1+i$, $b = 1+j$ and $z = i^2 = j^2$, then $ab+ba = ij(1+z)$ while $a^2 = b^2 = 1+z$. There is a further error later on when the matrix algebra is deformed to four copies of the field, since a non-commutative algebra can never have a flat deformation to a commutative algebra.

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